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Differential Structure of Greechie Logics

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A liaison between quantum logics and non-commutative differential geometry is outlined: a class of quantum logics are proved to possess the structure of discrete differential manifolds. We show that the set of proper elements of an arbitrary atomic Greechie logic is naturally endowed by Koszul's differential calculus.

INTRODUCTION

In this paper we explore a liaison between quantum logic and noncommutative geometry. Namely, we show that there is a class of quantum logics which carries a natural differential structure.

It was established by Koszul (1960) that differential calculus on smooth manifolds admits a purely algebraic reformulation in terms of graded differential modules over algebras of smooth functions. Several versions of noncommutative geometry—operator extensions of classical (sometimes called commutative) geometry—stem from Koszul's formalism. Geometrical models based on finite-dimensional algebras have been of particular interest (Baehr *et al.*, 1995; Zapatrin, 1997) due to their possible relevance to "empirical quantum geometry." This research resulted in the formalism of *discrete differential manifolds*—finite sets whose algebra of functions is endowed with a "differential envelope." It was proved that many classical geometrical features (Dimakis and Müller-Hoissen, 1998) survive in these models.

This paper is organized as follows. In Section 1 we associate (following Rota, 1968) with an arbitrary poset P a noncommutative algebra Ω called the incidence algebra of P. Then we show that if P is "good enough," the algebra Ω acquires some useful properties, for instance, becomes graded. In Section 2 we give a brief outline of Koszul's (1960) formalism of the calculus of differentials and introduce the general notion of discrete differential mani-

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fold. In Section 3 we introduce a class of posets, called differentiable, which can be treated as discrete differential manifolds, and, finally, show that atomic Greechie logics are always differentiable.

1. INCIDENCE ALGEBRAS

1.1. Algebras of Scalars in Dirac's Notation

Let P be a set. Denote by $\mathcal{H}$ the space of all finite formal linear combinations of elements of P written as Dirac's ket vectors:

$$\mathcal{H} = \operatorname{span}\{|p\rangle\}_{p \in P} \tag{1}$$

and by $\mathcal{H}^*$ the dual to $\mathcal{H}$ spanned on the basis of bra vectors:

$$\mathcal{H}^* = \operatorname{span}\{\langle q|\}_{q \in P}$$

such that

$$\langle p|q\rangle = \delta_{pq} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Now consider the set of symbols $|p\rangle\langle p|$ for all $p \in P$ and its linear span

$$\mathcal{A} = \operatorname{span}\{|p\rangle\langle p|\}_{p \in P} \tag{3}$$

and endow it with the operation of multiplication

$$|p\rangle\langle p| \cdot |q\rangle\langle q| = |p\rangle\langle p|q\rangle\langle q| = \begin{cases} |p\rangle\langle p| & \text{if} \quad p = q \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

making $\mathcal{A}$ an associative and commutative algebra.

We have defined $\mathcal{A}$ as an algebra of formal symbols (3). However, the elements of $\mathcal{A}$ can be treated as operators on both $\mathcal{H}$,

$$(|p\rangle\langle p|)|q\rangle := |p\rangle\langle p|q\rangle = \begin{cases} |p\rangle & \text{if} \quad p = q \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{H}^*$,

$$\langle q|(|p\rangle\langle p|) := \langle q|p\rangle\langle p| = \begin{cases} \langle p| & \text{if} \quad p = q \\ 0 & \text{otherwise} \end{cases}$$

1.2. Incidence Algebras of Posets and Their Moduli of Differentials

The notion of the incidence algebra of a poset was introduced by Rota (1968) in a purely combinatorial context. Let P be a partially ordered set: $P = (P, \leq)$. Take all ordered pairs p, q of elements of P, form the linear hull

$$\Omega = \operatorname{span}\{|p\rangle\langle q|\}_{p \leq q} \tag{5}$$

and extend the formula (4) to define the product in Ω:

$$|p\rangle\langle q| \cdot |r\rangle\langle s| = |p\rangle\langle q|r\rangle\langle s| = \langle q|r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & \text{if} \quad q = r \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

One may doubt the correctness of this definition of the product: who guarantees that $|p\rangle\langle s|$ is still in Ω when $q = r$? But recall that P is partially ordered: $|p\rangle\langle q| \in \Omega$, $|q\rangle\langle s| \in \Omega$ means $p \leq q$ and $q \leq s$, therefore $p \leq s$, that is why $|p\rangle\langle s| \in \Omega$. The obtained algebra Ω with the product (6) is called the *incidence algebra* of the poset P.

The incidence algebra Ω is obviously associative, but not commutative in general. The algebra $\mathcal{A}$ of scalars is a maximal commutative subalgebra of Ω.

Let us split Ω considered as a *linear space* rather than an algebra into two subspaces

$$\Omega = \mathcal{A} \oplus \mathcal{R}$$

and call

$$\mathcal{R} = \operatorname{span}\{|p\rangle\langle q|\}_{p < q}$$

the *module of differentials* of the poset P. In fact, it follows directly from (6) that for any $a \in \mathcal{A}$ and any $\omega \in \mathcal{R}$ both $a\omega$ and ωa are in $\mathcal{R}$. It also follows directly from (6) that for any $a, b \in \mathcal{A}$, $\omega \in \mathcal{R}$,

$$(a\omega)b = a(\omega b)$$

Therefore the module of differentials $\mathcal{R}$ is always $\mathcal{A}$-bimodule.[2]

1.3. Incidence Algebras of Jordan–Hülder Posets

Recall that a poset P is said to satisfy the *Jordan–Hölder condition* (Birkhoff, 1967) if, for any ordered pair $p, q \in P$, $p < q$, the lengths of all maximal chains $p < r < \ldots < s < q$ are equal. In this case with every basic element $|p\rangle\langle q|$ of Ω the following nonnegative integer can be associated:

$$\deg|p\rangle\langle q| = \text{the length of a maximal chain between } p \text{ and } q \tag{7}$$

splitting Ω into linear subspaces

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \ldots \tag{8}$$

with

[2] Although $\mathcal{A}$ is commutative, $a\omega \neq \omega a$ in general.

$$\Omega^0 = \mathrm{span}\{|p\rangle\langle p|\} = \mathcal{A}$$

$$\Omega^n = \mathrm{span}\{|p\rangle\langle q|\}_{\deg|p\rangle\langle q|=n}$$

making Ω a graded algebra:

$$\forall\omega \in \Omega^m, \quad \omega' \in \Omega^n \qquad \omega\omega' \in \Omega^{m+n}$$

and therefore making the module of differentials $\mathcal{R}$ a graded $\mathcal{A}$-bimodule:

$$\mathcal{R} = \Omega^1 \oplus \Omega^2 \oplus \ldots$$

2. GRADED DIFFERENTIAL ALGEBRAS

An algebraic version of differential calculus on manifolds due to Koszul (1960) is presented in this section. It admits powerful generalizations which gave rise, in particular, to noncommutative geometry [in the Dubois-Violette version; see Djemai (1995) for an outline].

Let $\mathcal{M}$ be a smooth manifold. Denote by $\mathcal{A}$ the algebra of smooth functions on $\mathcal{M}$ and by $\mathcal{T}^*$ the cotangent bundle over $\mathcal{M}$:

$$\mathcal{A} = C^\infty(\mathcal{M}); \qquad \mathcal{T} = T^*(\mathcal{M})$$

The elements of the exterior product $\mathcal{T}^* \wedge \ldots \wedge \mathcal{T}^*$ are called differential forms. Denote

$$\Omega^0 = \mathcal{A} \qquad \text{the space of scalars}$$

$$\Omega^1 = \mathcal{T}^* \qquad \text{the space of 1-forms}$$

$$\Omega^n = \wedge_n\mathcal{T}^* \qquad \text{the space of } n\text{-forms}$$

and form the direct sum

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \ldots$$

which is a graded algebra with respect to the exterior product $\wedge$ of differential forms:

$$\forall\omega \in \Omega^m, \quad \omega' \in \Omega^n \qquad \omega \wedge \omega' \in \Omega^{m+n}$$

The module of differentials

$$\mathcal{R} = \Omega^1 \oplus \Omega^2 \oplus \ldots$$

is a graded $\mathcal{A}$-bimodule. In classical differential geometry $\forall a \in \mathcal{A}$, $\omega \in \Omega^n$, $a\omega = \omega a$.

An important operator defined on Ω makes it a *differential calculus*. This is the Cartan differential D: $\Omega \to \Omega$, which has the following properties:

$$
\begin{aligned}
D(\Omega^m) &\subseteq \Omega^{m+1} \\
D^2 &= 0 \\
D\mathbf{1} &= 0 \quad \text{(where } \mathbf{1} \text{ is the unit element of the algebra } \mathcal{A}\text{)} \\
\forall \omega \in \Omega^m, \omega' \in \Omega^n \quad & D(\omega \wedge \omega') = D\omega \wedge \omega' + (-1)^m \omega \wedge \omega'
\end{aligned}
\tag{9}
$$

In general, a graded algebra Ω built from an algebra $\mathcal{A}$ endowed with an operator D satisfying (9) is called a *differential calculus* over the algebra $\mathcal{A}$.

The entire differential structure of the manifold $\mathcal{M}$ is captured in its differential algebra Ω. For instance, the space of vector fields is the dual to Ω^1.

Differential calculi over finite-dimensional commutative algebras were thoroughly studied in the last decade. As a result of this research an analog of (pseudo-) Riemannian geometry on finite and discrete sets was built (Baehr *et al.*, 1995; Dimakis and Müller-Hoissen, 1998). The triples

$$(P, \Omega(P), D)$$

are referred to as *discrete differential spaces*, where P is a set (at most countable), and $(\Omega(P), D)$ is a graded differential algebra over the algebra $\mathcal{A}$ of scalars on P.

We show that if we take a Greechie logic $\mathcal{L}$, then the order structure on $\mathcal{L}$ induces (in unambiguous way!) a differential calculus making it a discrete differential space.

3. DIFFERENTIAL CALCULI ON GREECHIE LOGICS

3.1 Differentiable Posets

A poset P is said to be *differentiable* if it (a) possesses the Jordan–Hölder property (see Section 1.3) and (b) admits an operator d on the space $\mathcal{H}$ of scalars on P, d: $\mathcal{H} \to \mathcal{H}$ such that

$$
\begin{aligned}
d^2 &= 0 \\
\deg|dp\rangle\langle q| &= \deg|p\rangle\langle q| + 1
\end{aligned}
\tag{10}
$$

where deg is the degree of elements of the incidence algebra $\Omega(P)$ defined as in (7): let $d|p\rangle = \Sigma\, \epsilon_s|s\rangle$; then $|dp\rangle\langle q|$ denotes the sum of only such $\epsilon_s|s\rangle$ for which $|s\rangle\langle q| \in \Omega$.

When P is a differential poset we are always in a position to define the operator D: $\Omega \to \Omega$ as follows:

$$D(|p\rangle\langle q|) := |dp\rangle\langle q| - (-1)^{\deg|p\rangle\langle q|}|p\rangle\langle qd| \tag{11}$$

where $\langle qd|$ is the action of the adjoint to d operator on bra vectors: see (15) below.

We claim that the operator D has the properties of a Cartan differential. Verify the conditions (9) for D. Let $|p\rangle\langle q| \in \Omega^m$ then

$$\deg|dp\rangle\langle q| = \deg|p\rangle\langle q| + 1$$

according to (10). To verify the second condition, calculate the value of $D\mathbf{1}$ on an arbitrary basic vector $|r\rangle$ of $\mathcal{H}$. Since $\mathbf{1} = \Sigma|p\rangle\langle p|$ we have

$$D\mathbf{1}|r\rangle = \left(\sum_{p\in P} D|p\rangle\langle p|\right)|r\rangle = \sum_{p\in P} |dp\rangle\langle p|r\rangle - \sum_{p\in P} |p\rangle\langle pd|r\rangle$$

$$= |dr\rangle - \left(\sum_{p\in P} |p\rangle\langle p|\right)|dr\rangle = |dr\rangle - \mathbf{1}|dr\rangle = 0$$

To verify the third condition (9), let $\omega = |p\rangle\langle q|$ with $\deg|p\rangle\langle q| = m$ and let $\omega' = |r\rangle\langle s|$ with $\deg|r\rangle\langle s| = n$. Then

$$D(\omega\omega') = \langle q|r\rangle D|p\rangle\langle s| = \langle q|r\rangle(|dp\rangle\langle s| - (-1)^{m+n}|p\rangle\langle sd|)$$

On the other hand, we have

$$\begin{aligned} D\omega \cdot \omega' &= (|dp\rangle\langle q| - (-1)^m|p\rangle\langle qd|)|r\rangle\langle s| \\ &= \langle q|r\rangle|dp\rangle\langle s| - \langle q|d|r\rangle(-1)^m|p\rangle\langle s| \\ \omega \cdot D\omega' &= |p\rangle\langle q|(|dr\rangle\langle s| - (-1)^n\langle q|r\rangle|p\rangle\langle sd|) \end{aligned}$$

Therefore

$$\begin{aligned} D\omega \cdot \omega' + (-1)^m\omega \cdot D\omega' &= \langle q|r\rangle(|dp\rangle\langle s| - (-1)^{m+n}|p\rangle\langle sd|) \\ &= D(\omega\omega') \end{aligned}$$

So, we conclude that any differential poset becomes a discrete differential manifold whenever a border operator d, (10), is specified.

3.2. Differential Structure on Simplicial Complexes

A *simplicial complex* $\mathcal{K} = (\mathcal{K}, V)$ is a collection $\mathcal{K}$ of of nonempty subsets (called *simplices*) of a set V (called the set of *vertices* of $\mathcal{K}$) such that

$$\forall s, s' \subseteq V \qquad s \in \mathcal{K} \quad \text{and} \quad s' \subseteq s \quad \text{imply} \quad s' \in \mathcal{K}$$

In particular, a simplex is a complex consisting of *all* nonempty subsets of the set of its vertices.

Any simplicial complex $\mathcal{K}$ consists of simplices which are, in turn, sets. That is why $\mathcal{K}$ is partially ordered by set inclusion. With any simplex p a positive integer $\#p$ is associated from the cardinality of p considered set:

$$\#p = \mathrm{card}\{v \in V: v \in p\} - 1$$

Consider the incidence algebra $\Omega = \Omega(\mathcal{K})$ (Section 1.2) of the complex $\mathcal{K}$. With any $|s\rangle\langle t| \in \Omega$ we associate

$$\deg|s\rangle\langle t| := \#s - \#t \tag{12}$$

making the algebra Ω graded:

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \ldots$$

In any simplicial complex the border operator d is defined

$$d|p\rangle = \sum \epsilon_s|s\rangle \tag{13}$$

with $\epsilon_s = \pm 1$ such that

$$d^2 = 0$$

$$\forall v \in V \quad d|v\rangle = 0 \tag{14}$$

$$\text{if} \quad \#p = m, \quad \text{then for any } s \text{ from (13)} \quad \#s = m - 1$$

The adjoint to d, called the *coborder operator*, acts in $\mathcal{H}^*$. Due to Dirac's notation we may use the same symbol d for both border and coborder operators with no confusion:

$$d: \quad \langle p| \mapsto \langle pd|$$

so that

$$\langle p|dq\rangle = \langle pd|q\rangle = \langle p|d|q\rangle \tag{15}$$

Let us verify the conditions (10) for an arbitrary simplicial complex $\mathcal{K}$. The first condition (10) follows from the first condition (14) and the second condition (10) follows from (12) and the third condition of (14); therefore:

- Any simplicial complex $\mathcal{K}$ is a differentiable poset.
- The border operator on $\mathcal{K}$ makes the set of its simplices a discrete differential manifold.

3.3. The Differential Structure on Atomic Greechie Logics

An atomic σ-orthomodular poset $\mathcal{L}$ is called an *atomic Greechie logic* if it can be represented as a union of almost disjoint Boolean algebras:

$$\mathcal{L} = \cup_i \mathcal{B}_i$$
$$\mathcal{B}_i \cap \mathcal{B}_j = \left[\begin{array}{l}\{\mathbf{0}, \mathbf{1}\} \\ \{\mathbf{0}, \mathbf{1}, v, v'\}\end{array}\right. \tag{16}$$

where v is an atom of $\mathcal{L}$.

Let $\mathcal{L}$ be a Greechie logic with the set of atoms V. In this section we show that the poset P of *proper* elements of $\mathcal{L}$,

$$P := \mathcal{L} \backslash \{\mathbf{0}, \mathbf{1}\} \tag{17}$$

is differentiable and build the border operator on P making it a discrete differential manifold.

Let us build the simplicial complex $\mathcal{K} = (\mathcal{K}, V)$ starting from the decomposition (16) of $\mathcal{L}$. The set V of atoms of $\mathcal{L}$ will be the set of vertices of $\mathcal{K}$. A nonempty subset $s \subseteq V$ will be a simplex of $\mathcal{K}$ whenever s is a *proper* (sic!) subset of atoms of a block $\mathcal{B}_i$ of $\mathcal{L}$:

$$\mathcal{K} := \{s \subseteq V: \exists \mathcal{B}_i \; s \subset V(\mathcal{B}_i),\, s \neq \mathbf{0},\, V(\mathcal{B}_i)\}$$

where $V(\mathcal{B}_i)$ is the set of atoms of the block $\mathcal{B}_i$. The poset P is Jordan–Hölder. To prove it, let $p, q \in P$, $p \neq q$; then (since they are proper elements of $\mathcal{L}$) there is a unique block $\mathcal{B}_i$ from (16) which contains them. We put

$$\deg |p\rangle\langle q| := \#_i\, q - \#_i p \tag{18}$$

as in (12) with

$$\#_i p = \operatorname{card} \{v \in \mathcal{B}_i: v \leq p\}$$

With every simplex $s \in \mathcal{K}$ an element of the poset P, (17), can be associated. Take the mapping f from $\mathcal{K}$ to $\mathcal{L}$

$$f(s) := \bigvee\nolimits_{\mathcal{L}} \{v \in V: v \in s\} \tag{19}$$

which is surjective (since any element of P is contained in a block and thus can be expressed as a join of atoms), but not injective (since an element of P can belong to more than one block).

Extend the mapping (19) to $f: \mathcal{H}(\mathcal{K}) \to \mathcal{H}(P)$ by linearity and introduce the border operator on $\mathcal{H}(P)$:

$$d|p\rangle := \sum_{f(s)=p} |f(ds)\rangle \tag{20}$$

and verify the conditions (10). First calculate its square:

$$d^2|p\rangle = \sum_{f(s)=p} d|f(ds)\rangle = \sum_{f(s)=p} d \sum \epsilon_{t_s} |f(t_s)\rangle = \sum_s f(|dds\rangle) = 0$$

The second condition (10) follows from (18) since for any component of the right-hand side of the sum

$$|dp\rangle\langle q| = \sum_{s:f(s)=p} |f(ds)\rangle\langle q|$$

is in a unique block in accordance with (16).

So, the set P of all proper elements of an arbitrary atomic Greechie logic becomes a discrete differential manifold.

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